The Coolest Thing in Pre-Calculus: Polar Complex Numbers and Euler's Formula

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Abstract

From a crushing defeat in a high school math contest, I learned how a number is like a transformation of points in the complex plane, and came to appreciate what's probably the coolest equation in analysis, "Euler's formula":

$$e^{\mathbf{i}\phi} = \cos\phi + \mathbf{i}\sin\phi$$

This paper recounts what I learned on my way there, and should be within the reach of an advanced high school student.

This isn't a real mathematical paper. Terminology isn't precise.

If you understand the addition formulas for $\sin x$ and $\cos x$ you'll understand most of my math; if you're also familiar with the Taylor series (the Maclaurin series in particular), all of this paper is within your grasp.

The Contest Problem

This was the final problem from a high school math contest in the 1970s:

> Find two distinct numbers, not 0 or 1, such that each is the square of the other.

In other words, find two numbers x and y such that:

$$x = y^2 \tag{1}$$

$$y = x^2 \tag{2}$$

$$x, y \notin \{0, 1\} \tag{3}$$

Solution. Once you see it, the answer is simplicity itself. The key is not to assume, as I did that day, that $x, y \in \mathbb{R}$; they're complex numbers.

Substitute (2) into (1) to yield $x = x^4$ or more canonically:

$$x^4 - x = 0 \tag{4}$$

Factoring yields

$$x(x^3 - 1) = 0$$

and

$$x(x-1)(x^2+x+1) = 0 \tag{5}$$

If any factor of (5)—*i.e.* x or $(x-1)$ or (x^2+x+1)—is 0, then (4) is satisfied. But (3) means that only the third factor can be 0, *i.e.*, that the solution must be a root of

$$x^2 + x + 1 = 0$$

which we solve using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Since $a = b = c = 1$, the solutions are

$$x = \frac{-1 \pm \sqrt{-3}}{2}$$

or, writing $\mathbf{i}$ for the positive square root of -1:

$$x \in \left\{ \frac{-1 + \mathbf{i}\sqrt{3}}{2}, \frac{-1 - \mathbf{i}\sqrt{3}}{2} \right\}$$

And y is whichever one x isn't.

Checking it. Let's consider the solution $x = \frac{-1+\mathbf{i}\sqrt{3}}{2}$, $y = \frac{-1-\mathbf{i}\sqrt{3}}{2}$. Does x^2 truly equal y?

$$x^2 = \frac{1 - 2\mathbf{i}\sqrt{3} - 3}{4} = \frac{-1 - \mathbf{i}\sqrt{3}}{2} = y$$

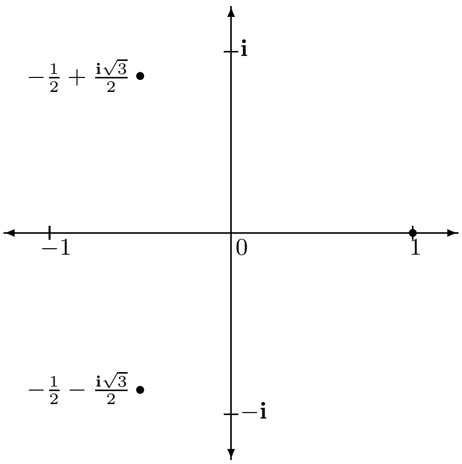
Yes. Likewise, we can see that $y^2 = x$:

$$y^2 = \frac{1 + 2\mathbf{i}\sqrt{3} - 3}{4} = \frac{-1 - \mathbf{i}\sqrt{3}}{2} = x$$

And $x^3 = x^2 x$ which is

$$\left(\frac{-1 + \mathbf{i}\sqrt{3}}{2}\right)\left(\frac{-1 - \mathbf{i}\sqrt{3}}{2}\right) = \frac{(-1)^2 - (\mathbf{i}\sqrt{3})^2}{4}$$
$$= \frac{1+3}{4} = 1$$

And there we have it: the solutions are cube roots of 1. Let's draw them:



If we write them as ordered pairs corresponding to rectangular coordinates for points in the complex plane—*i.e.* (a, b) for $a + b\mathbf{i}$—we'll have a list like this:

- $(1, 0)$
- $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$
- $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$

We can write the *polar coordinates* for these points in the form (r, θ), where

$$a = r\cos\theta, b = r\sin\theta \qquad (0 \le \theta < 2\pi)$$ [1]

- $(1, 0)$
- $(1, \frac{2\pi}{3})$
- $(1, \frac{4\pi}{3})$

That these are cube roots of 1 is easier to see in the second list. Indeed, the points lie on the unit circle, equidistant from each other.

This led me to an astonishing discovery. Probably I found it in a book, but I don't think a big deal was ever made of it.

[1] Intuitively, θ is the angle between the X-axis, and a vector from $(0, 0)$ passing through the point in question. For $(1, 0)$, $\theta = 0$; for $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $\theta = {}^{2\pi}/_{3}$.

Multiplying the numbers is like adding the angles

If we multiply two complex numbers with polar coordinates (r, θ) and (s, ϕ), their product has polar coordinates $(r \cdot s, \theta + \phi)$. That is, you *multiply* the magnitudes but *add* the angles.

Informal proof. Translate (r, θ) and (s, ϕ) into rectangular coordinates:

$$(a, b) = (r\cos\theta, r\sin\theta) \tag{6}$$
$$(c, d) = (s\cos\phi, s\sin\phi) \tag{7}$$

Since

$$(a + b\mathbf{i}) \cdot (c + d\mathbf{i}) = ac - bd + (ad + bc)\mathbf{i}$$

the product's rectangular coordinates will be

$$(ac - bd, ad + bc) \tag{8}$$

Our task is to demonstrate that the point with rectangular coordinates (8) has the polar coordinates

$$(r \cdot s, \theta + \phi) \tag{9}$$

that is, to show

$$ac - bd = r \cdot s \cdot \cos(\theta + \phi) \tag{10}$$
$$ad + bc = r \cdot s \cdot \sin(\theta + \phi) \tag{11}$$

This isn't very hard. Recall the addition formulas

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{12}$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha \tag{13}$$

Substitute (6) and (7) into the left-hand side of (10) and factor:

$$\begin{aligned} ac - bd &= r\cos\theta \cdot s\cos\phi - r\sin\theta \cdot s\sin\phi \\ &= r \cdot s \cdot (\cos\theta\cos\phi - \sin\theta\sin\phi) && (14) \\ &= r \cdot s \cdot \cos(\theta + \phi) && (15) \end{aligned}$$

(Substitute (12) into (14) to derive (15).)

Similarly expanding (11), rearranging terms and substituting (13) yields:

$$\begin{aligned} ad + bc &= bc + da \\ &= r\sin\theta \cdot s\cos\phi + s\sin\phi \cdot r\cos\theta \\ &= r \cdot s \cdot (\sin\theta \cdot \cos\phi + \sin\phi \cdot \cos\theta) \\ &= r \cdot s \cdot \sin(\theta + \phi) && (16) \end{aligned}$$

which is what we set out to prove.

Now it is possible that $\theta + \phi \geq 2\pi$ though both are in in the interval $[0, 2\pi)$; if that happens, (15) and (16) will still be true if we use $(\theta + \phi - 2\pi)$ as the angle.

Geometrically, multiplying an arbitrary complex number by (r, θ) is like magnifying (or minifying) its vector by r, and rotating it by θ.

Complex numbers as matrices

The combinations above bear an interesting similarity to matrix multiplication. Basically we can consider a complex number $a + b\mathbf{i}$ to be isomorphic with a matrix

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Then

$$\begin{aligned} (a + b\mathbf{i}) \cdot (c + d\mathbf{i}) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cdot \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \\ &= \begin{bmatrix} ac - bd & -ad - bc \\ bc + ad & -bd + ac \end{bmatrix} \\ &= \begin{bmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{bmatrix} \\ &= (ac - bd) + (ad + bc)\mathbf{i} \qquad (17) \end{aligned}$$

From this we can see that a complex number whose polar coordinates are $(1, \theta)$ can be represented by the "rotation matrix":[2]

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & cos\theta \end{bmatrix}$$

The Wikipedia article[2] shows how the rotation matrix can effectively rotate a point in the xy-plane as follows:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} \qquad (18)$$

Note that in (18) we wrote the rotation matrix as a 2×2 matrix but the point (x, y) as a column vector. We can represent both the rotation matrix and the (x, y) point as complex numbers if we wish; rotation would then be isomorphic with complex multiplcation.

[2] https://en.wikipedia.org/wiki/Rotation_matrix downloaded 2015-02-25

$$\begin{aligned} &\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & cos\theta \end{bmatrix} \cdot \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \\ &\quad = \begin{bmatrix} x\cos\theta - y\sin\theta & -y\cos\theta - x\sin\theta \\ x\sin\theta + y\cos\theta & -y\sin\theta + x\cos\theta \end{bmatrix} \\ &\quad = \begin{bmatrix} x\cos\theta - y\sin\theta & -(x\sin\theta + y\cos\theta) \\ x\sin\theta + y\cos\theta & x\cos\theta - y\sin\theta \end{bmatrix} \end{aligned}$$

which is an exact match for (18), modulo the representation of a complex number as a 2×2 matrix.

In this case, the multiplication is commutative; the multiplication of a 2×2 matrix by a 2×1 column vector cannot be.

Euler's formula

We can also see that adding the angles is like multiplying the numbers if we rewrite the Maclaurin series[3] expansions of $\sin x$ and $\cos x$ and rearrange terms carefully.

Readers not familiar with the Maclaurin series (or the Taylor series) can nevertheless appreciate this section by taking on faith that $\sin x$ and $\cos x$ and e^x are in fact representable by the expansions offered here.[4]

First, $\sin x$ is expanded thus:

$$\begin{aligned} \sin x &= \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} \qquad (19) \end{aligned}$$

Since $-1 = \mathbf{i}^2$, we can rewrite (19) as:

$$\sin x = \sum_{n=0}^{\infty} \frac{\mathbf{i}^{2n} \cdot x^{2n+1}}{(2n+1)!}$$

so that

$$\mathbf{i}\sin x = \sum_{n=0}^{\infty} \frac{(\mathbf{i}x)^{2n+1}}{(2n+1)!} \qquad (20)$$

[3] http://mathworld.wolfram.com/MaclaurinSeries.html downloaded 2015-02-25

[4] The arguments for both sin and cos are in *radians*; one radian is the angle subtended by a circular arc whose length is equal to the radius. Thus a 90°-angle is $\pi/2$ radians, a 60°-angle is $\pi/3$ radians, and a full circle (360°) is 2π radians.

Why $\mathbf{i} \sin x$? We'll see in a minute. Similarly, we rewrite the Maclaurin series for $\cos x$

$$\begin{aligned} \cos x &= \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(\mathbf{i}x)^{2n}}{(2n)!} \end{aligned} \tag{21}$$

The Maclaurin series expansion for e^x is:

$$\begin{aligned} e^x &= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= \sum_{i=0}^{\infty} \frac{x^i}{i!} \end{aligned} \tag{22}$$

Now we're ready to combine (20) and (21) and rearrange terms:

$$\cos x + \mathbf{i} \sin x = \sum_{m=0}^{\infty} \frac{(\mathbf{i}x)^m}{m!} \tag{23}$$

$$= e^{\mathbf{i}x} \tag{24}$$

We substitute (22) into (23) to arrive at (24), which is known as "Euler's formula." More information on this formula is on Wikipedia[5] and Wolfram MathWorld[6].

The wikipedia article quotes Feynman as calling the equation "the most remarkable formula in mathematics." I can't disagree.

Logarithms.

In a complex number with polar coordinates (r, θ), what we're saying is that θ acts rather like a logarithm: the *product* of the complex numbers has an angle that's the *sum* of the angles of each factor. And the reason that's so is precisely because the angle is part of the number's algorithm: the "imaginary" part.

Thus, if a complex number has polar coordinates (r, θ), we can define $\zeta = \ln r$, provided that $r > 0$. Then the complex number's value is equal to

$$e^{\zeta + \mathbf{i}\theta} \tag{25}$$

Put diffrently, we can say that $\zeta + \mathbf{i}\theta$ is its logarithm (base e).

[5] https://en.wikipedia.org/wiki/Euler%27s_formula downloaded 2015-02-23

[6] http://mathworld.wolfram.com/EulerFormula.html downloaded 2015-02-23

Are not unique. You may recall that sin and cos are periodic functions, that in particular

$$\sin(x + 2\pi n) = \sin x$$

and

$$\cos(x + 2\pi n) = \cos x$$

for all integers n.

It follows therefore that the complex numbers represented by $(r, \theta + 2\pi n)$ for various values of n are in fact one and the same number; this means that whenever we refer to "the" logarithm of a complex number, we may as well say "$\pm 2\pi \mathbf{i} n$ for integer n."

The logarithm, therefore, isn't unique in $\mathbb{C}$. I believe this is usually solved by restricting θ as for example

$$\theta \in [0, 2\pi)$$

Conclusion

A real scientific or mathematical paper would have some profound conclusions here, but this is just my recounting of some great fun I had with math in high school. It's because of things like this that I studied math in college, and the reason I consider mathematics to be a fine art.